SIMULATION OF NONLINEAR BOUNDARY CONDITIONS

BY ELECTRICAL NETWORKS

R. A. Pavlovskii

UDC 681.142.334

The special features of the method of successive approximations as applied to the simulation of nonlinear boundary conditions by electrical networks are examined. The dependence of the convergence of the method on the initial data is analyzed. Results of an experimental test of the conclusions are presented.

The problem of the steady-state temperature distribution in a solid with convective heat removal at its boundary and a number of other potential theory problems reduce to finding a harmonic function ϑ satisfying a boundary condition of the form

$$\vartheta + f\left(\frac{\partial\vartheta}{\partial n}\right)\bigg|_{\Gamma} = 1. \tag{1}$$

Henceforth we shall assume that the function f is defined for $\partial\vartheta/\partial n > 0$, is twice continuously differentiable, is positive ($f > 0$), and $f(0) = 0$.

One effective method of solving Laplace's equation with the boundary condition (1) involves simulation by networks of electrical resistances [1, 2]. Condition (1) is represented by additional external resistances R connected to the boundary junctions as shown in Fig. 1.

In this scheme the potential U_i of the i-th boundary junction is related to the current I_i to that junction by the relation

$$U_i + \Delta U_i(I_i) = C, \quad i = 1, 2, \ldots, m. \tag{2}$$

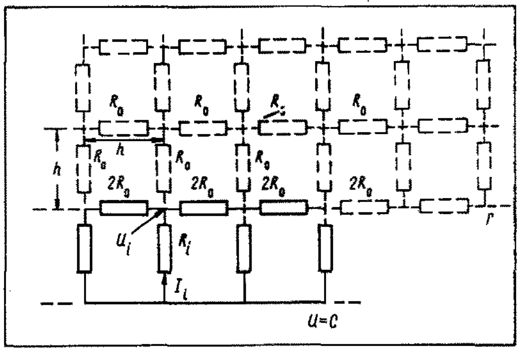

Fig. 1. Electrical network for simulating boundary condition (1).

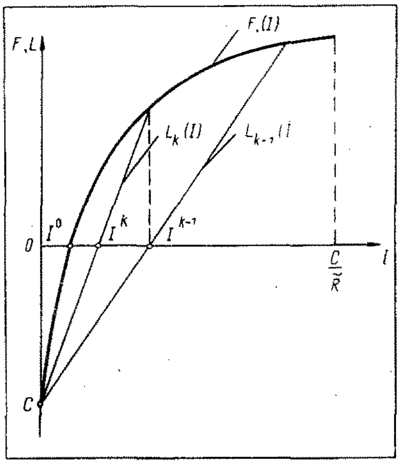

Fig. 2. Graphical interpretation of successive current approximations.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 5, pp. 918-925, November, 1969. Original article submitted November 26, 1968.

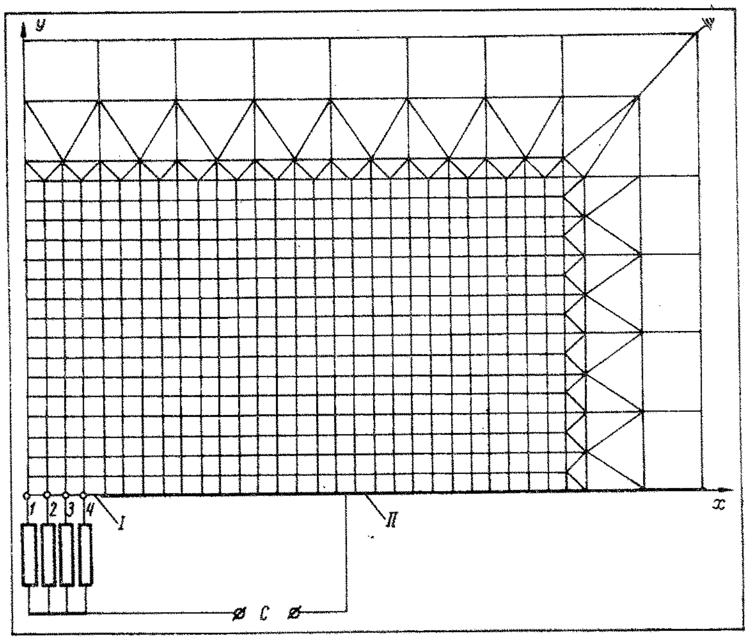

Fig. 3. Schematic diagram of electrical network for region under consideration.

For a network of uniform steps the relation between the thermal and electrical quantities is given by

$$\frac{\vartheta}{U}=\frac{f\left(\frac{\partial\vartheta}{\partial n}\right)}{\Delta U_i\,(I_i)}=\frac{1}{C},\quad \frac{\partial\vartheta}{\partial n}=\frac{R_0N_0}{C}\,I_i. \tag{3}$$

When the parameters of the body are independent of the temperature distribution the function f is linear

$$f\left(\frac{\partial\vartheta}{\partial n}\right)=\frac{1}{\mathrm{Bi}}\,\frac{\partial\vartheta}{\partial n}$$

and (1) goes over into a boundary condition of the third kind. This is simulated by connecting constant external resistances of identical magnitudes R_i to all the boundary junctions of the network

$$R_i=\frac{N_0R_0}{\mathrm{Bi}}=\mathrm{const},\quad i=1,\ 2,\ \ldots,\ m.$$

The problem is much more complicated when the function f is not linear. In this case the magnitudes of the external resistances must be chosen so that the following nonlinear relation between the potential drop and the current density is satisfied for all boundary junctions:

$$\Delta U=Cf\left(\frac{R_0N_0}{C}\,I\right)=\varphi\,(I). \tag{4}$$

This can be achieved either by connecting nonlinear two-terminal networks satisfying Eq. (4) to the boundary junctions of the network [3, 4], or by using the method of successive approximations [5, 6].

The method of nonlinear two-terminal networks is very effective but requires developing supplementary functional units for the simulator. The method of successive approximations is simpler technically but in many cases is not a convergent process. We determine how the convergence of the method of successive approximations depends on the form of the function f.

The method of successive approximations can be applied to electrical networks in two forms. The first of these is based on the successive refinement of the values of the currents to the junctions (current approximation), and the second on the refinement of the values of the potential drops across the external resistances (voltage approximation). We consider each of these modifications in turn.

TABLE 1. Results of Simulation by Method of Successive Approximations

Version No.			I				II				III			
Number of junctions			1	2	3	4	1	2	3	4	1	2	3	4
Current approximation (I in mA and R in Ω)	1	R	600	300	300	300	600	300	300	300	600	300	300	300
		I	5,2	10,6	11,7	16,5	5,2	10,6	11,7	16,5	5,2	10,6	11,7	16,5
	2	R	540	270	250	190	410	210	510	2000	1030	570	490	430
		I	5,3	11	12,8	24,5	7,0	15	7,9	3,0	4,10	8,10	8,90	12,8
	3	R	540	260	230	140	2400	1300	90	60	1100	550	540	480
		I	5,2	11	13	32	1,4	2,35	21	52	3,85	7,80	8,35	11,6
	4	R	540	260	230	110	—	—	—	—	1120	550	540	490
		I	5,1	10,7	12,5	38					3,85	7,80	8,35	11,5
	5	R	540	260	240	100	—	—	—	—	—	—	—	—
		I	5,1	10,8	12,1	41								
	6	R	540	260	240	100	—	—	—	—	—	—	—	—
		I	5,1	10,8	12,1	41								
Voltage approximation (ΔU in V and R in Ω)	1	R	600	300	300	300	1000	500	500	500	600	300	300	300
		ΔU	3,12	3,18	3,51	4,95	4,0	4,1	4,35	5,5	3,12	3,18	3,51	4,95
	2	R	360	170	130	20	720	370	390	480	1280	640	610	530
		ΔU	1,62	1,60	1,38	1,60	3,50	3,60	3,95	5,35	4,30	4,40	4,70	5,55
	3	R	1140	570	690	570	650	330	360	460	1120	550	540	490
		ΔU	4,2	4,3	4,7	5,6	3,30	3,45	3,85	5,30	4,25	4,30	4,50	5,55
	4	R	120	50	30	20	620	320	350	460	1120	550	540	490
		ΔU	0,70	0,65	0,50	1,50	3,30	3,40	3,85	5,35	4,25	4,30	4,50	5,55
	5	R	2000	1000	1000	620	620	320	350	460	—	—	—	—
		ΔU					3,30	3,40	3,85	5,35				

The method of successive current approximations consists in the following. As a first approximation identical external resistances R_i^1, chosen either arbitrarily* or from physical considerations, are connected to all the boundary junctions of the network. The currents I_i^1 are then measured for each of the junctions and these measured values are used in (4) to find the refined values of the external resistances

$$R_i^2 = \frac{\varphi(I_i^1)}{I_i^1}. \tag{5}$$

In the second approximation resistances R_i^2 are connected to the boundary junctions of the network and the steps described above are repeated. This process of successive approximations is continued (if it converges) until the values of I_i in two successive approximations coincide within the required accuracy for all boundary junctions simultaneously.

It is impossible to give an analysis of the convergence of the process in the general case of an arbitrary boundary of the region. Let us consider, therefore the simplest case of a single boundary junction ($m = 1$) for which we write Eq. (2) in the form

$$\tilde{R}I + \varphi(I) = C, \tag{6}$$

where $\tilde{R}$ is the resistance between the boundary junction and the zero potential junctions.

We regard the boundary condition (6) as an equation for the current

$$F(I) = 0, \tag{7}$$

where $F(I) = \tilde{R}I + \varphi(I) - C$. It is easy to see that the function F is defined on the interval $I \in [0, C/\tilde{R}]$.

In using the method of successive current approximations we essentially approximate F at each (k-th) step by a linear function L_k

$$F(I) \sqsupset L_k(I) = (\tilde{R} + R^k)\, I - C, \tag{8}$$

i.e., the experimental determination of the current to a junction in each approximation is equivalent to solving the linear equation $L_k(I) = 0$. In the above scheme for finding the external resistances the equation

$$R^k = \frac{F(I^{k-1}) + C}{I^{k-1}} - \tilde{R}, \tag{9}$$

is satisfied. This leads to the following recurrence relation for the currents:

*In particular one may choose $R_i^1 = 0$ ($i = 1, 2, \ldots m$).

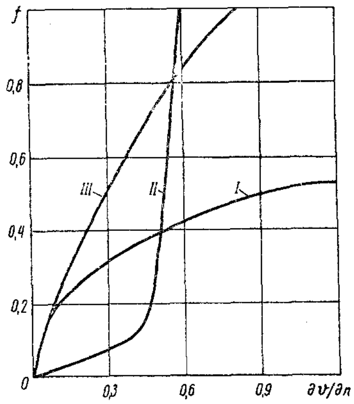

Fig. 4. Graphs of functions appearing in boundary condition (29).

$$I^k = \frac{C}{F(I^{k-1}) + C} I^{k-1}. \tag{10}$$

In this interpretation the successive approximations scheme agrees with the familiar method of chords (Fig. 2). The recurrence relation (10) also agrees with the corresponding recurrence relation for the method of chords if the points $I = 0$ and $I = I^{k-1}$ are regarded as interpolation points. Consequently the convergence conditions for the method of chords can be used directly to estimate the convergence of the method of successive approximations for boundary condition (6) on an electrical network [7].

According to [7] the sequence I^k will converge if the function $F(I)$ satisfies the Fourier conditions: a) the first and second derivatives F' and F'' do not change sign in the interval $I \in [0, C/R]$; b) at an interpolation point which is a common end of all chords (the point $I = 0$ in Fig. 2) the inequality

$$F(0)F''(0) > 0. \tag{11}$$

is satisfied.

The following estimate of the rate of convergence can be obtained*:

$$|I^0 - I^k| \leqslant \frac{H}{h} |I^k - I^{k-1}|,$$

where $h \leq |F'(I)| \leq H$ for $I \in [0, C/\tilde{R}]$.

Equation (4) shows that the derivatives of the auxiliary function F and the original function f are connected by the relations

$$F' = \tilde{R} + R_0 N_0 f', \quad F'' = \frac{R_0^2 N_0^2}{C} f''. \tag{12}$$

Since $F_0 = -C$ it follows from (11) and (12) that the sufficient conditions for the convergence in the method of successive current approximations are that

$$f' \geqslant 0^{\dagger}, \tag{13}$$

$$f'' < 0. \tag{14}$$

hold in the interval $d\vartheta/dn \in [0, R_0N_0/\tilde{R}]$.

If conditions (13) and (14) are not satisfied the sequence I^k may converge, diverge, or oscillate about the value I^0. For example, for $f' \geq 0$ and $f'' > 0$ the convergence of the process, according to [7], is ensured only if the inequality

$$\frac{F'(I^0)\, I^0}{C} < 2. \tag{15}$$

is satisfied. It is easy to see that the following estimates are valid:

$$I^0 \leqslant \frac{C}{\tilde{R}}, \quad F'(I^0) \leqslant F'\left(\frac{C}{\tilde{R}}\right) = \tilde{R} + \varphi'\left(\frac{C}{\tilde{R}}\right). \tag{16}$$

By substituting (16) into (15) and using (4) we obtain a condition which is sufficient for a priori estimates of the convergence of the current approximation when $f'' > 0$:

$$f'\left(\frac{R_0 N_0}{\tilde{R}}\right) \leqslant \frac{\tilde{R}}{R_0 N_0}. \tag{17}$$

However, as will be shown below, when $f'' > 0$ the method of successive voltage approximations is more suitable than the current approximation.

The method of successive voltage approximations is based on the use of the relation which is the inverse of (4):

*More accurate convergence estimates are given in [7].

†It is easy to see that condition (13) is more rigid than the condition that F' does not change sign.

$$I = \bar{\varphi}(\Delta U) = \frac{C}{R_0 N_0} \bar{f}\left(\frac{\Delta U}{C}\right), \tag{18}$$

where $\bar{f}$ is the inverse of f.*

The scheme of the method differs from that considered earlier in that at each step measurements are made of the potential drops across the external resistances (ΔU_i) rather than the currents to the junctions. In this case the values of the external resistances are refined by using (18):

$$R_i^k = \frac{\Delta U_i^{k-1}}{\bar{\varphi}(\Delta U_i^{k-1})}. \tag{19}$$

By considering a single junction of the network and using the same procedure as in the estimate of the convergence of the successive current approximations method it is easy to show that the present problem is reduced to finding the root ΔU^0 of the transcendental equation

$$\Phi(\Delta U) = 0, \tag{20}$$

where $\Phi(\Delta U) = \tilde{R}\bar{\varphi}(\Delta U) + \Delta U - C$ is a function defined on the interval $\Delta U \in [0, C]$. Finding the root of Eq. (20) by successive approximations is related to approximating the function Φ at each (k-th) step by a linear function M_k:

$$\Phi(\Delta U) \sqsupset M_k(\Delta U) = \left(\frac{\tilde{R}}{R^k} + 1\right)\Delta U - C.$$

The magnitudes of the potential drops in two successive approximations are connected by the recurrence relation

$$\Delta U^k = \frac{C}{\Phi(\Delta U^{k-1}) + C} \Delta U^{k-1}, \tag{21}$$

which is analogous to Eq. (10). Thus in the present case, as before, the analogy between the method of successive approximations and the method of chords can be used.

The derivatives of Φ are:

$$\Phi' = 1 + \frac{\tilde{R}}{R_0 N_0} \bar{f}', \tag{22}$$

$$\Phi'' = \frac{\tilde{R}}{R_0 N_0 C} \bar{f}''. \tag{23}$$

If f is strictly monotonic the relations between the derivatives of the direct and inverse functions are given by the well-known expressions

$$\bar{f}' = \frac{1}{f'}, \tag{24}$$

$$\bar{f}'' = -\frac{f''}{(f')^3}. \tag{25}$$

Since $\Phi(0) = -C$, by starting with the Fourier conditions and using (22)-(25) we obtain the following sufficient conditions for the convergence in the successive voltage approximations method:

$$f' \geqslant 0, \tag{26}$$

$$f'' > 0. \tag{27}$$

If $f' \geq 0$ but $f'' < 0$, it can be shown by using an inequality of the type (15) and the maximum estimates for ΔU^0 and $\Phi'(\Delta U^0)$ that the convergence of the voltage approximation method is ensured only when the following sufficient condition is satisfied

$$\bar{f}'(1) \leqslant \frac{R_0 N_0}{\tilde{R}}. \tag{28}$$

As noted above, these results apply only to the simplest case of a single network junction. There is reason to suppose, however, that qualitatively these results remain valid also in the general case of an arbitrary boundary approximated by m network junctions.

*As is well known, the function $\bar{f}$ exists if f is strictly monotonic. Henceforth we assume that this is the case.

To test this assumption we investigated the two-dimensional problem of finding a harmonic function ϑ on a half-plane for the boundary conditions (Fig. 3)

$$\text{I)}\ \vartheta + f\left(\frac{\partial\vartheta}{\partial y}\right) = 1, \quad 0 \leqslant |x| < 1,\ y = 0; \tag{29}$$

$$\text{II)}\ \vartheta = 0, \quad 1 < |x| < \infty, \quad y = 0. \tag{30}$$

This problem was solved by simulating the field on a type ÉI-12 electro-integrator having a network with an "expander" [8] designed to approximate an infinite region. Since the system under consideration is symmetric about the y axis, half the region (Fig. 3) was set up on the network.

The function f was specified graphically. Three versions were investigated (Fig. 4).

I) $f'' < 0$, gently sloping curve;

II) $f'' > 0$;

III) $f'' < 0$, steep curve.

Simulation was performed for the following model parameters: C = 7.0 V, $R_0 = 100\,\Omega$, $N_0 = 3.5$ steps.

Boundary condition (29) was simulated for each of the indicated versions by the methods of successive current approximations and voltage approximations. The results are shown in Table 1.

Table 1 shows that for curve I the process of successive current approximations converges but the voltage approximation diverges; for curve II the voltage approximation converges but the current approximation diverges; for curve III both approximations converge. The last result arises from the fact that curve III is much steeper than curve I (cf. (28)).

It is easy to see that the data obtained correspond to the previously formulated conditions for the convergence in the method of successive approximations.

NOTATION

ϑ	is the function sought (the temperature);
Bi	is the Biot number;
f, φ, F, Φ, L and M	are the symbols for functions;
Γ	is the boundary surface;
n	is the outward normal to the boundary surface;
U	is the electric potential;
ΔU	is the potential drop across an external resistance;
C	is a constant value of the potential;
I	is the current to a boundary junction of the electrical network;
R_0	is the resistance between adjacent junctions of the electrical network;
R	is the external resistance at a boundary junction;
$\tilde{R}$	is the resistance between a boundary junction and "infinity";
N_0	is the number of steps of the network corresponding to the scale of length of the original system;
m	is the number of boundary junctions of the network.

LITERATURE CITED

1. W. J. Karplus, Analog Simulation: Solution of Field Problems, McGraw–Hill, New York (1958).
2. L. A. Kozdoba, Inzh. Fiz. Zh., 11, 6, 809 (1966).
3. Yu. M. Matsevityi, in: Heat and Mass Transfer [in Russian], Vol. 8, Nauka i Tekhnika, Minsk (1968), p. 448.
4. M. Sh. Gladshtein and L. I. Kadaner, in: Electric Fields in Electrolytes [in Russian], Nauka, Novosibirsk (1967), p. 39.
5. N. P. Gnusin, Zh. Prikl. Khim., 34, 10, 2275 (1961).
6. N. P. Gnusin and A. I. Maslii, in: Electric Fields in Electrolytes [in Russian], Nauka, Novosibirsk (1967), p. 78.
7. A. M. Ostrowski, Solutions of Equations and Systems of Equations, Academic Press, New York (1960).
8. L. V. Nitsetskii, Uchenye zapiski Rizhskogo politekhnicheskogo instituta, 12, No. 2 (1965).